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ON THE LATTICE PROPERTY OF THE PLANE AND SOME PROBLEMS OF DIRAC, MOTZKIN AND ERDŐS IN COMBINATORIAL GEOMETRY

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Dedicated to Paul Erdős on his seventieth birthday

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Let S be a set of n non-collinear points in the Euclidean plane. It will be shown here that for some point of S the number of *connecting lines* through it exceeds $c \cdot n$. This gives a partial solution to an old problem of Dirac and Motzkin. We also prove the following conjecture of Erdős: If any straight line contains at most $n-x$ points of S, then the number of connecting lines determined by S is greater than $c \cdot x \cdot n$.

1. Introduction

G. A. Dirac [3] and T. S. Motzkin [11], independently of each other and at the same time, proposed the following problem: Is it true that in every non-collinear set of n points some point is connected to the others by at least $c \cdot n$ straight lines? In fact, they conjectured that this holds for $c = 1/2$. Here we show the validity of the first conjecture. Note that the strong form (i.e., $c = 1/2$) is certainly false for small values of n (see Grünbaum [7]). Naturally, it may well be that the strong Dirac—Motzkin conjecture is correct as soon as n is sufficiently large.

Now we state our result in a more explicit form. Let S be a set of non-collinear points in the Euclidean plane and consider the *connecting lines* determined by S, i.e., the straight lines passing through at least two points of S. For any point $P \in S$ let $f(S, P)$ denote the number of connecting lines through P. Let

$$f(n) = \min_{|S|=n} \max_{P \in S} f(S, P).$$

Here and in what follows $|S|$ denotes the number of elements of the set S. Our first result gives a partial answer to Dirac—Motzkin conjecture.

Theorem 1.1. $f(n) > c_1 \cdot n$.

Throughout this paper $c_1, c_2, c_3, \ldots$ denote positive absolute constants.

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We have learned that E. Szemerédi and W. T. Trotter [12], independently of us and at the same time, have proved Theorem 1.1 using entirely different ideas. Actually, they proved a stronger result (see the remark at Theorem 1.5).

Observe that Theorem 1.1 immediately follows from Theorem 1.2 below (in fact, they are essentially equivalent apart from constant factors).

Theorem 1.2. *Let $L_1, L_2, \dots, L_t$ denote all possible connecting lines of the n-element coplanar point-set S. If $\max_{1 \leq i \leq t} |L_i \cap S| = n-x$ (x, $0 \leq x \leq n-2$ is arbitrary), then*

$$c_2 \cdot x \cdot n < t \leq x(n-x) + \binom{x}{2} + 1 \leq 1 + x \cdot n.$$

Note that the upper bound is almost trivial. It is left to the reader. Theorem 1.2 settles a conjecture of P. Erdős in the affirmative (see Erdős [4] and Erdős [6]). The particular case $x < c_3 \cdot n^{1/2}$ was completely solved much earlier by L. M. Kelly and W. Moser [9].

In the proofs of this paper we shall use only the following three combinatorial properties of the plane.

(α) *Almost disjointness:* Any two straight lines have at most one point in common.

(β) Given any family $L_1, L_2, \dots, L_r$ of concurrent straight lines, then there is a natural *circular order* among the lines L_i.

(γ) Let there be given two families $L_1', L_2', \dots, L_r'$ and $L_1'', L_2'', \dots, L_r''$ of concurrent straight lines in the plane. These $2r$ lines partition the plane into disjoint convex regions. Simple geometric consideration shows that any straight line L intersects at most $2r+1$ of these regions. Indeed, the $2r$ intersection-points $\{L \cap L_i'\}$, $\{L \cap L_j''\}$, $(1 \leq i, j \leq r)$ divide L into at most $2r+1$ intervals. We shall refer to this fact as the *lattice property* of the plane.

From this it follows that our method is "flexible" enough and works for other classes of curves, too. For example, our argument gives without any modification the analogous to Theorem 1.2 result for *pseudolines*, i.e., simple closed (in the projective plane) curves each two having exactly one point in common at which they "cross": In any arrangement of pseudolines having n vertices in which each pair is joined by a pesudoline, if there are at most $n-x$ vertices on the same pseudoline, then the number fo pseudolines exceeds $c_2 \cdot x \cdot n$.

Using the analogy between *unit circles* (i.e., circles with common radius one) and straight lines, it is not hard to modify our argument to give the following result.

Theorem 1.3. *Given any n-element coplanar point-set of diameter less than two (i.e., any two points determine exactly two unit circles), then the number of distinct unit circles containing at least two of the points is always greater than $c_4 \cdot n^2$.*

Of course, Theorem 1.3 remains true for arbitrary point-sets with a constant factor depending on the diameter only.

We cannot resist mentioning here a related conjecture of Erdős [5] which probably needs a completely new idea: Given n points in the plane, then the number of unit circles containing at least *three* of the points is $o(n^2)$.

Our method also works to give a partial answer to a problem of E. Jucovič [8]: Given any set S of points in the upper half-plane, then denote by $g(S)$ the number of circles containing at least two points of S and touching the x-axis (in a degenerated case we allow straight lines parallel to the x-axis). Let

$$g(n) = \min_{|S|=n} g(S).$$

Twenty years ago Jucovič asked for bounds on $g(n)$. The following result shows that $g(n)$ is essentially as large as possible.

Theorem 1.4. $g(n) > c_5 \cdot n^2$.

Consider the Poincaré model of the *hyperbolic* plane (see e.g. Coxeter [2]). The points of the hyperbolic plane are interpreted as inner points of the Euclidean upper half-plane determined by the x-axis.

Horocycles (i.e., curves of identically one curvature) are either circles of the upper half-plane touching the x-axis or straight lines parallel to the x-axis. So every couple of points determines two horocycles. Now Theorem 1.4 says that every set of n points in the hyperbolic plane determines at least $c_5 \cdot n^2$ distinct horocycles.

Let $t(S, k)$ denote the number of straight lines containing at least k points of S. Let

$$t(n, k) = \max_{|S|=n} t(S, k),$$

i.e., $t(n, k)$ denotes the largest integer for which there is a set of n points in the plane for which there are $t(n, k)$ lines each containing at least k of the points.

Let $t^*(S, k)$ denote the number of straight lines containing at least k but less than $2k$ points of S. Similarly, let

$$t^*(n, k) = \max_{|S|=n} t^*(S, k).$$

Clearly

$$t^*(S, k) \leqq t(S, k) \quad \text{and} \quad t^*(n, k) \leqq t(n, k).$$

Both Theorem 1.1 and Theorem 1.2 will be deduced from the following estimate.

Theorem 1.5. $t^*(n, k) \leqq c_6 \cdot \dfrac{n^2}{k^{2+\delta}}$

with $\delta = 1/20$ *for all* $2 \leqq k \leqq (2n)^{1/2}$.

In the proof we shall apply the "coordinate method" of Beck and Spencer [1].

We have to mention here that Szemerédi and Trotter [12] proved the bound $t(n, k) < c \cdot n^2 \cdot k^{-3}$ for $2 \leqq k \leqq n^{1/2}$. The $n^{1/2} \times n^{1/2}$ lattice shows that their estimate is best possible apart from a constant factor (we leave the trivial proof to the reader.)

Finally, we remark that a wide range of beautiful problems in combinatorial geometry can be found in Erdős' survey paper [4] and W. Moser's summary [10]. Theorem 1.1 also yields a positive answer to Problem 18(a) and (b) in Moser [10]. For further consequences of Theorem 1.5., see the last section of this paper.

2. Proof of Theorem 1.5

We start with some trivial upper bounds on $t(n, k)$.

Lemma 2.1.

$$t(n,k) \leqq \frac{\binom{n}{2}}{\binom{k}{2}} \quad \textit{for} \quad 2 \leqq k \leqq n; \tag{1}$$

$$t(n,k) < \frac{2n}{k} \quad \textit{for} \quad (2n)^{1/2} < k \leqq n. \tag{2}$$

Proof. Let S be an n-element point-set with $t(S, k) = t(n, k)$. Counting the point-pairs of S in two different ways we obtain

$$t(n,k)\cdot\binom{k}{2} \leqq \binom{n}{2},$$

and (1) follows.

To prove (2) we apply the following general observation: If $L_1, L_2, \ldots, L_t$ are subsets of the n-element set S such that $|L_i| \geqq k$ $(1 \leqq i \leqq t)$, $|L_i \cap L_j| \leqq 1$ $(1 \leqq i \neq j \leqq t)$ and $k > (2n)^{1/2}$, then $t < 2n/k$. Indeed, the assumption $t = \lceil 2n/k \rceil$ (upper integral part) leads to a contradiction as follows:

$$n = |S| \geqq \left|\bigcup_{i=1}^{t} L_i\right| \geqq \sum_{i=1}^{t} |L_i| - \sum_{1 \leqq i < j \leqq t} |L_i \cap L_j|$$

$$\geqq \sum_{i=1}^{t} |L_i| - \binom{t}{2} \geqq t\cdot k - \binom{t}{2} > 2n - \binom{(2n)^{1/2}}{2} > n. \quad \blacksquare$$

Next we need

Lemma 2.2. *Let there be given n subsets $S_i \subset S$, $1 \leqq i \leqq n$ where $|S| = n$ and $\sum_{i=1}^{n} |S_i| > \vartheta \cdot n^2$ $(0 < \vartheta < 1)$. Then there exist an index $i_0 \in [1, n]$, an index-set $I(i_0) \subset [1, n] \setminus \{i_0\}$ with $|I(i_0)| \geqq \frac{3}{n(n-2)}\binom{\vartheta n}{3}$ and for every $i_1 \in I(i_0)$ there exists an index-set $I(i_0, i_1) \subset [1, n] \setminus \{i_0, i_1\}$ with $|I(i_0, i_1)| \geqq \frac{3}{2n(n-1)}\binom{\vartheta n}{3}$ such that*

$$|S_{i_0} \cap S_{i_1} \cap S_{i_2}| \geqq \frac{3}{2(n-1)(n-2)}\binom{\vartheta n}{3}$$

for all possible triples (i_0, i_1, i_2) satisfying the requirements $i_1 \in I(i_0)$ and $i_2 \in I(i_0, i_1)$.

Proof. For $1 \leqq r \leqq n$ let d_r be the number of sets S_i containing P_r where $S = \{P_1, P_2, \ldots, P_n\}$. Then

$$\frac{1}{n}\sum_{i=1}^{n} |S_i| = \frac{1}{n}\sum_{r=1}^{n} d_r > \vartheta \cdot n,$$

and hence

$$\sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \sum_{\substack{k=1 \\ k\neq j,i}}^{n} |S_i \cap S_j \cap S_k| = \sum_{r=1}^{n} d_r(d_r-1)(d_r-2)$$

$$\geqq n\cdot \bar{d}(\bar{d}-1)(\bar{d}-2) > n\cdot \vartheta n(\vartheta n-1)(\vartheta n-2)$$

where $\bar{d}=\left(\sum_{r=1}^{n} d_r\right)/n$.

Thus, there must exist an index $i_0\in[1,n]$ such that

$$\sum_{\substack{j=1 \\ j\neq i_0}}^{n} \sum_{\substack{k=1 \\ k\neq j,i_0}}^{n} |S_{i_0} \cap S_j \cap S_k| > \vartheta n(\vartheta n-1)(\vartheta n-2). \tag{3}$$

Let J denote the set of indices $j\in[1,n]\setminus\{i_0\}$ for which

$$\sum_{\substack{k=1 \\ k\neq j,i_0}}^{n} |S_{i_0} \cap S_j \cap S_k| \leqq \frac{1}{2(n-1)}\cdot \vartheta n(\vartheta n-1)(\vartheta n-2).$$

Since

$$\sum_{j\in J} \sum_{\substack{k=1 \\ k\neq j,i_0}}^{n} |S_{i_0} \cap S_j \cap S_k| \leqq (n-1)\cdot \frac{1}{2(n-1)}\cdot \vartheta n(\vartheta n-1)(\vartheta n-2) \tag{4}$$

$$= \frac{1}{2}\vartheta n(\vartheta n-1)(\vartheta n-2),$$

by (3) and (4) we get for $I(i_0)=[1,n]\setminus(J\cup\{i_0\})$,

$$\frac{1}{2}\vartheta n(\vartheta n-1)(\vartheta n-2) < \sum_{i_1\in I(i_0)} \sum_{\substack{k=1 \\ k\neq i_0,i_1}}^{n} |S_{i_1} \cap S_{i_0} \cap S_k| \leqq |I(i_0)|\cdot(n-2)\cdot n.$$

Consequently, $|I(i_0)| > \frac{3}{n(n-2)}\binom{\vartheta n}{3}$.

For each $i_1\in I(i_0)$ let $K(i_1)$ denote the set of indices $k\in[1,n]\setminus\{i_0,i_1\}$ for which

$$|S_{i_0} \cap S_{i_1} \cap S_k| \leqq \frac{1}{4(n-1)(n-2)}\vartheta n(\vartheta n-1)(\vartheta n-2).$$

Since

$$\sum_{k\in K(i_1)} |S_{i_0} \cap S_{i_1} \cap S_k| \leqq (n-2)\cdot \frac{1}{4(n-1)(n-2)}\vartheta n(\vartheta n-1)(\vartheta n-2)$$

$$= \frac{1}{4(n-1)}\vartheta n(\vartheta n-1)(\vartheta n-2),$$

and by definition

$$\sum_{\substack{k=1 \\ k\neq i_0,i_1}}^{n} |S_{i_0} \cap S_{i_1} \cap S_k| > \frac{1}{2(n-1)}\vartheta n(\vartheta n-1)(\vartheta n-2)$$

if $i_1 \in I(i_0)$, we obtain that

$$\frac{1}{4(n-1)} \vartheta n(\vartheta n-1)(\vartheta n-2) < \sum_{i_2 \in I(i_0, i_1)} |S_{i_0} \cap S_{i_1} \cap S_{i_2}| \leqq |I(i_0, i_1)| \cdot n$$

where $I(i_0, i_1) = [1, n] \setminus (K(i_1) \cup \{i_0, i_1\})$. Hence $|I(i_0, i_1)| > \vartheta n(\vartheta n-1)(\vartheta n-2)/4n \cdot (n-1)$, which completes the proof of the lemma. ∎

We shall prove Theorem 1.5 with a sufficiently large constant factor c_6 (e.g. $c_6 = 2^{1000}$ will be an appropriate choice). Let $S = \{P_1, P_2, \ldots, P_n\}$ be an n-element point-set in the Euclidean plane with the property $t^*(S, k) = t^*(n, k)$. That is, there are $t^*(n, k)$ straight lines L_j, $1 \leqq j \leqq t^*(n, k)$ such that each of them contains at least k but less than $2k$ points of S. We shall refer to the lines L_j as *k-lines* of S.

Let $t^*(n, k) = y \cdot n^2 \cdot k^{-2-\delta}$. We shall show that $y \leqq c_6$ where c_6 is a sufficiently large absolute constant independent of n and k (note that, by (1), $y \leqq 2k^\delta$).

For every $P_i \in S$ let $L_{i(1)}, L_{i(2)}, \ldots, L_{i(t_i)}$ denote the k-lines passing through P_i. Throw away the point P_i from each line $P_{i(j)}$, $1 \leqq j \leqq t_i$. Then we obtain two open half-lines $L'_{i(j)}$ and $L''_{i(j)}$. We may assume that $|L'_{i(j)} \cap S| \geqq |L''_{i(j)} \cap S|$. Since $|L'_{i(j)} \cap S| + |L''_{i(j)} \cap S| \geqq k-1$, we have $|L'_{i(j)} \cap S| \geqq (k-1)/2$.

Starting from P_i select the first $\left\lceil \frac{k-1}{2} \right\rceil$ (upper integral part) points of S along the half-line $L'_{i(j)}$. Let $K_{i(j)}$ denote this $\left\lceil \frac{k-1}{2} \right\rceil$-element subset of $L'_{i(j)} \cap S$. We shall refer to $K_{i(j)}$ as a "*segment*" of type i.

Let us define the "star" S_i $(1 \leqq i \leqq n)$ as follows

$$S_i = \bigcup_{j=1}^{t_i} K_{i(j)}.$$

By a standard double-counting argument we have

$$\sum_{i=1}^{n} |S_i| \geqq t^*(n, k) \cdot k \cdot \left\lceil \frac{k-1}{2} \right\rceil > \frac{y}{4k^\delta} n^2.$$

Now applying Lemma 2.2 we obtain that

(5) there exist an index $i_0 \in [1, n]$, an index-set $I(i_0)$ with $|I(i_0)| > y^3 \cdot 2^{-10} \cdot n \cdot k^{-3\delta}$, and for every $i_1 \in I(i_0)$ there exists an index-set $I(i_0, i_1)$ with $|I(i_0, i_1)| > y^3 \cdot 2^{-10} \cdot n \cdot k^{-3\delta}$ such that $|S_{i_0} \cap S_{i_1} \cap S_{i_2}| > \frac{y^3}{2^{10} \cdot k^{3\delta}} n$ for all possible triplets (i_0, i_1, i_2) where $i_1 \in I(i_0)$ and $i_2 \in I(i_0, i_1)$.

We define a partition $S_i = S_i^{(1)} \cup S_i^{(2)} \cup \ldots \cup S_i^{(d)}$ $(1 \leqq i \leqq n)$ where the parameter $d (< k)$ will be fixed later. We recall that $L_{i(1)}, L_{i(2)}, \ldots, L_{i(t_i)}$ denote the k-lines passing through $P_i \in S$. Without loss of generality, we may assume that if $j < l$ then the slope of $L_{i(j)}$ is smaller than the slope of $L_{i(l)}$, i.e., $-\infty <$ slope $(L_{i(1)}) <$ slope $(L_{i(2)}) < \ldots <$ slope $(L_{i(t_i)}) < +\infty$ (we fix a pair of orthogonal coordinate axes). One can easily decompose the interval $[1, t_i]$ indo d subintervals $J_{i,1}, J_{i,2}, \ldots, \ldots, J_{i,d}$ such that for every set $S_i^{(\nu)} = \bigcup_{j \in J_{i,\nu}} K_{i(j)}$ we have

$$|S_i^{(\nu)}| \leqq \frac{n}{d}, \quad 1 \leqq \nu \leqq d. \tag{6}$$

The lattice property of the plane (see (γ) in Section 1) yields that the partition $S_i = \bigcup_{\nu=1}^{d} S_i^{(\nu)}$ has the following property:

(7) For any straight line L and for any two indices $i, j \in [1, n]$, L intersects at most $2d+1$ sets $S_i^{(\nu)} \cap S_j^{(\mu)}$ $(1 \leqq \nu, \mu \leqq d)$.

We distinquish two cases.

Case 1. For every $i_1 \in I(i_0)$ there exist an integer $q = q(i_1)$, $0 \leqq q \leqq \lceil \log d \rceil$, and a set $H_{i_1} \subset [1, d] \times [1, d]$ of index-pairs with $|H_{i_1}| \geqq d^2 \cdot 4^{-q}$ such that for all $(\nu, \mu) \in H_{i_1}$,

$$|S_{i_0}^{(\nu)} \cap S_{i_1}^{(\mu)}| \geqq 2^q \frac{n}{d^{2-3\delta}}.$$

We shall prove the impossibility of Case 1 by the trivial estimate (1). Fix $i_1 \in I(i_0)$ and $(\nu, \mu) \in H_{i_1}$. We recall that $S_{i_1}^{(\mu)}$ is the disjoint union of some segments of type i_1:

$$S_{i_1}^{(\mu)} = \bigcup_{j \in J_{i_1,\mu}} K_{i_1(j)}.$$

Reordering the indices we may assume that

$$S_{i_1}^{(\mu)} = K_{i_1(1)} \cup K_{i_1(2)} \cup \ldots \cup K_{i_1(r)} \quad (r = |J_{i_1,\mu}|).$$

Set $k_l = |S_{i_0}^{(\nu)} \cup K_{i_1(l)}|$ $(1 \leqq l \leqq r)$. Clearly $r \leqq 3n/dk$, since $r \cdot k/3 < r \cdot \dfrac{k-1}{2} \leqq \sum_{l=1}^{r} |K_{i_1(l)}| = |S_{i_1}^{(\mu)}| \leqq n/d$.

Thus, by hypothesis, we get

$$\frac{1}{r} \sum_{l=1}^{r} k_l = \frac{1}{r} |S_{i_0}^{(\nu)} \cap S_{i_1}^{(\mu)}| \geqq \frac{d \cdot k}{3n} \cdot 2^q \frac{n}{d^{2-3\delta}} > \frac{2^{q-2} \cdot k}{d^{1-3\delta}}. \tag{8}$$

Let

$$\Delta_q(h) = \Delta_q(h; \nu, \mu, i_1) = \frac{2^{h+q-2} \cdot k}{d^{1-3\delta}}, \quad h = 0, 1, 2, \ldots.$$

From (8) it follows the existence of an integer h', $0 \leqq h' \leqq O(\log d)$ such that

$$\sum_{l: \Delta_q(h'-1) < k_l \leqq \Delta_q(h')} k_l \geqq \frac{1}{2^{h'+2}} \sum_{l=1}^{r} k_l = \frac{1}{2^{h'+2}} |S_{i_0}^{(\nu)} \cap S_{i_1}^{(\mu)}|.$$

Hence

$$|\{l\in[1,r]\colon \Delta_q(h'-1)<k_l\leqq \Delta_q(h')\}|\geqq \frac{1}{\Delta_q(h')}\sum_{l:\,\Delta_q(h'-1)<k_l\leqq\Delta_q(h')} k_l$$

$$\geqq \frac{1}{\Delta_q(h')}\frac{1}{2^{h'+2}}|S_{i_0}^{(\nu)}\cap S_{i_1}^{(\mu)}|\geqq \frac{1}{\Delta_q(h')}\cdot\frac{1}{2^{h'+2}}\cdot 2^q\cdot\frac{n}{d^{2-3\delta}}=\frac{n}{4^{h'}\cdot k\cdot d}.$$

This means that there exist at least $n\cdot 4^{-h'}\cdot k^{-1}\cdot d^{-1}$ segments of type i_1, $K_{i_1(l)}$ such that

$$|S_{i_0}^{(\nu)}\cap K_{i_1(l)}|>\Delta_q(h'-1)=\frac{2^{q+h'-3}\cdot k}{d^{1-3\delta}}.$$

Repeating the same argument for *all* pairs $(\nu,\mu)\in H_{i_1}$, by the pigeonhole principle we conclude that for some fixed $h^*=h^*(i_1)$, $0\leqq h^*=O(\log d)$ there are at least

$$\frac{|H_{i_1}|}{c_7\log d}\cdot\frac{n}{4^{h^*}\cdot k\cdot d}\geqq\frac{d^2\cdot 4^{-q}}{c_7\log d}\cdot\frac{n}{4^{h^*}\cdot k\cdot d}=c_8\cdot 4^{-q-h^*}\cdot\frac{d\cdot n}{k\cdot\log d}$$

segments of type i_1 each of them containing $>\Delta_q(h^*-1)$ elements of one of the sets $S_{i_0}^{(\nu)}$, $1\leqq\nu\leqq d$.

Summing for $i_1\in I(i_0)$, again by the pigeonhole principle there must exist integers $\bar q$ and $\bar h$ with $0\leqq\bar q=O(\log d)$, $0\leqq\bar h=O(\log d)$ such that for at least $|I(i_0)|/c_9\cdot(\log d)^2$, indices $i_1\in I(i_0)$, $q(i_1)=\bar q$ and $h^*(i_1)=\bar h$. Since any k-line L_j contains less than $2k$ points of S, there are less than $2k$ segments with the *same* support L_j.

Summarising, we conclude that there exist at least (see also (5))

$$\frac{1}{2k}\cdot\frac{|I(i_0)|}{c_9(\log d)^2}\cdot\left(c_8\cdot 4^{-\bar q-\bar h}\cdot\frac{dn}{k\cdot\log d}\right)\geqq\frac{1}{2k}\cdot\frac{y^3\cdot 2^{-10}\cdot n\cdot k^{-3\delta}}{c_9(\log d)^2}\cdot\left(c_8\cdot 4^{-\bar q-\bar h}\cdot\frac{d\cdot n}{k\cdot\log d}\right)$$

$$=c_{10}\cdot y^3\cdot n^2\cdot d\cdot 4^{-\bar q-\bar h}\cdot k^{-2-3\delta}(\log d)^{-3}$$

distinct k-lines L_j each of them containing $>\Delta_{\bar q}(\bar h-1)=2^{\bar q+\bar h-3}\cdot k\cdot d^{-1+3\delta}$ points of one of the sets $S_{i_0}^{(\nu)}$, $1\leqq\nu\leqq d$. This yields

(9) $$c_{10}\cdot y^3n^2\cdot d\cdot 4^{-\bar q-\bar h}\cdot k^{-2-3\delta}\cdot(\log d)^{-3}\leqq\sum_{\nu=1}^{d}t(S_{i_0}^{(\nu)},\bar k)$$

where $\bar k=\Delta_{\bar q}(\bar h-1)$. Using (6) and (1) we obtain

(10) $$\sum_{\nu=1}^{d}t(S_{i_0}^{(\nu)},\bar k)\leqq d\cdot\frac{n^2}{d^2}\cdot(4^{-\bar q-\bar h+3}k^{-2}d^{2-6\delta})=4^3n^2d\cdot 4^{-\bar q-\bar h}k^2d^{6\delta}.$$

From (9) and (10) we conclude that

(11) $$y^3\cdot d^{6\delta}\cdot(\log d)^{-3}\leqq c_{11}\cdot k^{3\delta}.$$

The parameter d will be defined later so that (11) will lead to the desired upper bound $y\leqq c_6$.

Case 2. Assume that for some $i_1 \in I(i_0)$, the number of pairs (ν, μ) such that

$$|S_{i_0}^{(\nu)} \cap S_{i_1}^{(\mu)}| \geqq 2^q \frac{n}{d^{2-3\delta}}$$

is less than $d^2 \cdot 4^{-q}$ for every q, $0 \leqq q \leqq [\log d]$.

We recall that for every $i_2 \in I(i_0, i_1)$ we have (see (5)):

$$|\bar{S} \cap S_{i_2}| > \frac{y^3 \cdot 2^{-10}}{k^{3\delta}} n \quad \text{where} \quad \bar{S} = S_{i_0} \cap S_{i_1}.$$

Since S_{i_2} is the disjoint union of segments of type i_2 and each of these segments contains exactly $\left[\frac{k-1}{2}\right]$ points of S, it follows that there must exist $\geqq 2^{-10} y^3 k^{-1-3\delta} \cdot n$ segments $K_{i_2(l)}$ of type i_2 such that

$$|K_{i_2(l)} \cap \bar{S}| > \frac{y^3 2^{-11}}{k^{3\delta}} \cdot \left[\frac{k-1}{2}\right] \quad (i_2 \in I(i_0, i_1)).$$

Since any k-line L_j contains less than $2k$ points of S, there are less than $2k$ segments with the *same* support L_j. Combining these facts we obtain the existence of at least (see (5))

$$\frac{1}{2k} \cdot |I(i_0, i_1)| \cdot \frac{2^{-10} y^3}{k^{1+3\delta}} \cdot n \geqq 2^{-21} \cdot y^6 \cdot k^{-2-6\delta} \cdot n^2 \tag{12}$$

distinct k-lines L_j such that

$$|\bar{S} \cap L_j| > \frac{2^{-11} \cdot y^3}{k^{3\delta}} \cdot \left[\frac{k-1}{2}\right].$$

We need the following double-counting lemma.

Lemma 2.3. *Let* $S(\nu, \mu) \subset S$, $1 \leqq \nu, \mu \leqq d$ *be subsets with the properties:*

$$S(\nu, \mu) \cap S(\nu^*, \mu^*) = \emptyset \quad \text{if} \quad (\nu, \mu) \neq (\nu^*, \mu^*); \tag{13}$$

$$|S(\nu, \mu)| \leqq 2^q \cdot m \quad \textit{for all but less} \tag{14}$$

than $d^2 4^{-q}$ *pairs* (ν, μ), $q = 0, 1, 2, \ldots, [\log d]$.

Let $\bar{S} = \bigcup_{\nu=1}^{d} \bigcup_{\mu=1}^{d} S(\nu, \mu)$. *Assume that*

$$|\bar{S}| \leqq n. \tag{15}$$

Moreover, suppose that

(16) *any straight line* L *intersects at most* $2d+1$ *of the sets* $S(\nu, \mu)$, $1 \leqq \nu, \mu \leqq d$.

Then

$$t(\bar{S}, 4d) < \frac{1}{d}(3d^2 \cdot \log d \cdot m^2 + n \cdot m/2)$$

where $t(\bar{S}, 4d)$ denotes the number of straight lines containing at least $4d$ points of $\bar{S}$ (i.e., the number of $(\geqq 4d)$-lines of $\bar{S}$).

Remark. Hypothesis (14) immediately implies that $|S(v, \mu)| < 2d \cdot m$ for *all* pairs (v, μ).

Proof. Consider the quadruples (r_1, r_2, v, μ) such that $r_1 < r_2$ and for some $(\geqq 4d)$-line L of $\bar{S}$, $\{Q_{r_1}, Q_{r_2}\} \subset S(v, \mu)$ where $L \cap \bar{S} = \{Q_1, Q_2, Q_3, \ldots\}$. We count the number of these quadruples in two different ways.

Let L be a $(\geqq 4d)$-line in $\bar{S}$ where $L \cap \bar{S} = \{Q_1, Q_2, \ldots, Q_p\}$, $p \geqq 4d$. Let $Q_r \in S_{i_0}^{(v_r)} \cap S_{i_1}^{(\mu_r)}$, $1 \leqq r \leqq p$. From hypothesis (16) it follows by elementary calculation that the number of repetitions among the pairs (v_r, μ_r) $1 \leqq r \leqq p$ is greater than d. Thus, $d \cdot t(\bar{S}, 4d)$ gives a lower bound to the number of quadruples (r_1, r_2, v, μ).

Since any two distinct points determines exactly one straight line, we obtain that the expression

$$\sum_{v=1}^{d} \sum_{\mu=1}^{d} \binom{|S(v, \mu)|}{2}$$

gives an *upper* bound to the number of quadruplets (r_1, r_2, v, μ).

Summarising

$$t(\bar{S}, 4d) < \frac{1}{d} \sum_{v=1}^{d} \sum_{\mu=1}^{d} \binom{|S(v, \mu)|}{2}. \tag{17}$$

By (13) and (14) we have

$$\sum_{\substack{(v, \mu) \\ |S(v, \mu)| \leqq m}} \binom{|S(v, \mu)|}{2} \leqq \frac{n}{m} \cdot \binom{m}{2} < \frac{n \cdot m}{2}. \tag{18}$$

Furthermore, (14) yields that for every integer q, $0 \leqq q \leqq \lceil \log d \rceil$,

$$\sum_{2^q m < |S(v, \mu)| \leqq 2^{q+1} m} \binom{|S(v, \mu)|}{2} \leqq \binom{2^{q+1} m}{2} \cdot d^2 \cdot 4^{-q}. \tag{19}$$

Since $|S(v, \mu)| < 2^p m$ with $p = \lceil \log d \rceil$, by (17), (18) and (19) we get

$$t(\bar{S}, 4d) < \frac{1}{d} \left\{ \frac{n \cdot m}{2} + \sum_{q=0}^{\lceil \log d \rceil} \binom{2^{q+1} m}{2} \cdot d^2 \cdot 4^{-q} \right\}$$

$$\leqq \frac{1}{d} \left\{ \frac{n \cdot m}{2} + 3 \log d \cdot m^2 \cdot d^2 \right\},$$

which completes the proof of Lemma 2.3. ∎

Applying Lemma 2.3 with $S(v, \mu) = S_{i_0}^{(v)} \cap S_{i_1}^{(\mu)}$, $m = n \cdot d^{-2+3\delta}$ and $d = \lceil 2^{-13} \cdot y^3 \cdot k^{1-3\delta} \rceil$ (see also (7)), we obtain that there are less than

$$\frac{1}{d} \left\{ 3 \log d \cdot m^2 \cdot d^2 + \frac{n \cdot m}{2} \right\} \leqq 4n^2 \log d \cdot d^{-3+6\delta} \tag{20}$$

distinct k-lines L_j such that

$$|\bar{S}\cap L_j| \geqq 4d > 2^{-11}\cdot y^3\cdot k^{-3\delta}\cdot\left\lceil\frac{k-1}{2}\right\rceil.$$

Comparing (20) and (12) we conclude that

$$y^6\cdot d^{3-6\delta}\cdot(\log d)^{-1} \leqq c_{12}\cdot k^{2+3\delta}. \tag{21}$$

Let $\delta=1/20$. Then from (11), (21) and from the choice $d=\lceil 2^{-13}\cdot y^3\cdot k^{1-3\delta}\rceil$ it follows by elementary calculation that in both cases y is bounded by a sufficiently large absolute constant c_6 independent of n and k. The proof of Theeorem 1.5 is complete. ▮

3. Proofs of Theorems 1.1-2

First we need the following result.

Theorem 3.1. *Let there be given a set S of n points in the Euclidean plane. Then either* (α) *some straight line contains $>n/100$ points of S, or* (β) *the number of distinct straight lines determined by S exceeds $c_{13}\cdot n^2$.*

Proof. Let $L_1, L_2, \ldots, L_t$ denote the connecting lines of S, and let $l_i=|L_i\cap S|$, $1\leqq i\leqq t$. Counting the pointpairs of S in two different ways we obtain

$$\binom{n}{2} = \sum_{i=1}^{t}\binom{l_i}{2}. \tag{22}$$

Divide the right-hand side sum into three parts:

$$\Sigma_1 = \sum_{2^{c_{14}}\leqq l_i<(2n)^{1/2}}\binom{l_i}{2},\quad \Sigma_2 = \sum_{(2n)^{1/2}\leqq l_i\leqq n/100}\binom{l_i}{2}$$

and

$$\Sigma_3 = \sum_{2\leqq l_i<2^{c_{14}}}\binom{l_i}{2}+\sum_{n/100<l_i\leqq n}\binom{l_i}{2}.$$

First we estimate Σ_1 from above. By Theorem 1.5

$$\Sigma_1 = \sum_{j\geqq n_{14}}\ \sum_{\substack{2^j\leqq l_i<2^{j+1}\\ l_i<(2n)^{1/2}}}\binom{l_i}{2} \leqq \sum_{\substack{j\geqq c_{14}\\ 2^j<(2n)^{1/2}}} t^*(S,2^j)\cdot\binom{2^{j+1}}{2} \leqq \tag{23}$$

$$\leqq \sum_{\substack{j\geqq c_{14}\\ 2^j<(2n)^{1/2}}} t^*(n,2^j)\cdot\binom{2^{j+1}}{2} \leqq \sum_{j\geqq c_{14}} c_6\frac{n^2}{2^{2j+j/20}}\binom{2^{j+1}}{2} \leqq 2c_6\cdot n^2\sum_{j\geqq c_{14}}2^{-j/20} < \frac{1}{4}\binom{n}{2}$$

if c_{14} is sufficiently large.

Second we give an upper bound to Σ_2.
By (2) in Lemma 2.1 we have

$$(24)\quad \Sigma_2 = \sum_{j\geqq 0}\ \sum_{\substack{2^j(2n)^{1/2}\leqq l_i<2^{j+1}(2n)^{1/2}\\ l_i\leqq n/100}} \binom{l_i}{2} \leqq \sum_{2^j\leqq n^{1/2}/100} t(S, 2^j(2n)^{1/2})\cdot\binom{2^{j+1}(2n)^{1/2}}{2}$$

$$< \sum_{\substack{j\geqq 0\\ 2^j\leqq n^{1/2}/100}} 2\frac{n}{2^j(2n)^{1/2}}\cdot\binom{2^{j+1}(2n)^{1/2}}{2} \leqq \sum_{\substack{j\geqq 0\\ 2^j\leqq n^{1/2}/100}} 2^{j+3}n^{3/2} < \frac{1}{4}\binom{n}{2}.$$

Combining (22), (23) and (24) we get $\Sigma_3 > \frac{1}{2}\binom{n}{2}$.

If some straight line contains $>n/100$ points of S, then we are done. In the opposite case

$$\sum_{2\leqq l_i<2^{c_{11}}} \binom{l_i}{2} > \frac{1}{2}\binom{n}{2},$$

that is,

$$t \geqq \sum_{i:2\leqq l_i<2^{c_{14}}} 1 > \frac{1}{2}\binom{n}{2}\cdot\binom{2^{c_{14}}}{2}^{-1} = c_{13}\cdot n^2.$$

The proof of Theorem 3.1 is complete. ▮

Proof of Theorem 1.1. If alternative (β) in Theorem 3.1 holds then we are done by a simple averaging argument. In the opposite case let L denote a straight line containing more than $n/100$ points. Selecting any point $P\notin L$ the lines determined by the point-pairs (P, Q) $Q\in L$ are clearly distinct. ▮

Proof of Theorem 1.2. Again if alternative (β) in Theorem 3.1 holds then we are ready. If $\max_{1\leqq i\leqq t} |L_i\cap S| = |L_{i_0}\cap S| = n-x > \frac{n}{100}$ then we choose an $\left[\frac{n}{100}\right]$-element subset Z of $S\setminus L_{i_0}$. Joining all possible pairs (P_i, Q_j) where $P_i\in L_{i_0}$ and $Q_j\in Z$ we obtain

$$\geqq \frac{x}{100}\cdot(n-x) - \binom{\left[\frac{x}{100}\right]}{2} \geqq \frac{x}{100}\left(n-x-\left[\frac{x}{200}\right]\right) \geqq \frac{x}{100}\cdot\frac{n}{200}$$

distinct connecting lines. ▮

4. Proofs of Theorems 1.3-4

Proof of Theorem 1.4. We call a circle in the upper half-plane *x-circle* if the x-axis is tangent to it. Given any point-set S in the upper half-plane then denote by $t_0^*(S, k)$ the number of x-circles R such that $k\leqq|R\cap S|<2k$ $(k\geqq 2)$. Set

$$t_0^*(n, k) = \sup_{|S|=n} t_0^*(S, k).$$

The proof is based on the following estimate.

Lemma 4.1. $t_0^*(n, k) \leqq c_{15} \frac{n^2}{k^{2+\delta}}$ *with* $\delta = \frac{1}{20}$ *for all* k, $2 \leqq k \leqq (2n)^{1/2}$.

We outline a *proof* of Lemma 4.1 referring to Section 2. Let $S = \{P_1, P_2, \ldots, P_n\}$ be an n-element subset of the upper half-plane with the property $t_0^*(S, k) = t_0^*(n, k)$. Let $R_1, R_2, \ldots, R_t$, $t = t_0^*(n, k)$ denote all possible x-circles R_j with $k \leqq |R_j \cap S| < 2k$. For every P_i let $R_{i(1)}, R_{i(2)}, \ldots, R_{i(t_i)}$ denote the subsequence of $R_1, \ldots, R_t$ containing P_i. For any $R_{i(j)}$ let Q_j denote the only common point of the x-axis and the circle $R_{i(j)}$ (point of contact). The line segment P_iQ_j splits $R_{i(j)}$ into two arcs $R_{i(j)}^+$ and $R_{i(j)}^-$ where "+" and "−" are defined according to the counter-clockwise and clockwise order along $R_{i(j)}$ starting from P_i. For every $j (1 \leqq j \leqq t)$ either $|(R_{i(j)}^+ \setminus \{P_i\}) \cap S| \geqq \frac{k-1}{2}$ or $|(R_{i(j)}^- \setminus \{P_i\}) \cap S| \geqq \frac{k-1}{2}$. Thus, for some index-set $J_i \subset [1, t_i]$ with $|J_i| \geqq t_i/2$, $|(R_{i(j)}^* \setminus \{P_i\} \cap S| \geqq \frac{k-1}{2}$ for each $j \in J_i$. Here $* = *(i) = +$ or $-$ identically for all $j \in J_i$. Observe that the arcs $R_{i(j)}^* \setminus \{P_i\}$ $(j \in J_i)$ are pairwise disjoint. Starting from P_i select the first $\left\lceil \frac{k-1}{2} \right\rceil$ points of S along the arc $R_{i(j)}^* \setminus \{P_i\}$. Denote this $\left\lceil \frac{k-1}{2} \right\rceil$-element subset of $(R_{i(j)}^* \setminus \{P_i\} \cap S$ by $K_{i(j)}$. Finally, we define the "star" S_i $(1 \leqq i \leqq n)$ as follows

$$S_i = \bigcup_{j \in J_i} K_{i(j)}.$$

Now the proof of Lemma 4.1 proceeds along the same lines as that of Theorems 1.5. The only difference is that two points determine one straight line but two x-circles. This fact causes some change in the constant factors only. The desired "lattice property" for arcs is as follows: Let there be given a "compatible" family $A_1, A_2, \ldots, A_r$ of arcs in the upper half-plane each of them starting from a common point P and terminating at the x-axis. Here "compatibility" means that $A_i \cap A_j = \{P\}$ if $i \neq j$. Let $A_1', A_2', \ldots, A_r'$ be another compatible family of arcs in the upper half-plane with common starting point P' and with endpoints on the x-axis. These arcs A_i, A_j' partition the upper half-plane into disjoint regions. Now the "lattice property" says that any circle R intersects at most $4r$ of these regions. Indeed, the at most $4r$ intersections-points $R \cap A_i$, $R \cap A_j'$ divide R into at most $4r$ subarcs. ▌

Now the proof of Theorem 1.4 can be completed as follows. We say that an x-circle is determined by S if it contains at least two points of S. Repeating the standard double-counting argument of Section 3, one can easily deduce from Lemma 4.1 that either

(α) some x-circle contains more than $n/100$ points of S,

or

(β) the number of distinct x-circles determined by S exceeds $c_{16} \cdot n^2$.

If alternative (β) holds then we are done. In the opposite case denote by R_0 an x-circle containing $> n/100$ points of S. For any pair $P, P' \in R_0 \cap S$ $(P \neq P')$ denote by $R(P, P')$ the x-circle containing both P and P' and different from R_0. Thus we obtain the existence of more than $\binom{n/100}{2}$ distinct x-circles determined by S. ▌▌

Proof of Theorem 1.3. Exactly on the same line as that of Theorem 1.4 except the definition of the decomposition $R=R^+\cup R^-$. Let there be given a point $P\in S$ and a unit circle R with $P\in R$. The half-line starting from P and passing through the center of R splits R into two half-circles R^+ and R^-. ▮

5. Further conseq uences

We recall that $t(S, k)$ denotes the number of straight lines containing at least k points of S (e.g., $t(S, 2)$ denotes the number of connecting lines of S).

We shall deduce the following result from Theorem 1.2.

Theorem 5.1. *Given arbitrarily small $\varepsilon>0$, then there is a threshold $l=l(\varepsilon)$ depending on ε only such that for any non-collinear point-set S in the plane, $t(S, l)<\varepsilon\cdot t(S, 2)$.*

Proof.* Let $|S|=n$ and $\max\limits_L |L\cap S|=n-x$ $(x\geqq 1)$ where L is extended over all connecting lines of S. Then, by (1) we have

$$t(S, l)\leqq 1+t(S\setminus L, l-1)\leqq 1+\frac{\binom{x}{2}}{\binom{l-1}{2}}.$$

Now one can easily complete the proof, since by Theorem 1.2, $t(S, 2)>c_2\cdot x\cdot n$. ▮

Combining Theorem 1.2 and Theorem 5.1 we immediately obtain

Theorem 5.2.** *Let there be given n points in the plane and assume that any straight line contains at most $n-x$ of them. Then the number of connecting lines containing less than c_{17} of the points exceeds $\frac{1}{2}c_2\cdot x\cdot n$.* ▮

Next we state

Theorem 5.3. *Let there be given n points in the plane and assume that any circle (straight line) contains at most $n-y$ of them. Then the number of distinct circles (straight lines) containing at least three but not more than c_{17} of the points exceeds $c_{18}\cdot y\cdot n^2$.*

Proof. Let $S=\{P_1, P_2, \ldots, P_n\}$. Applying *inversions* with center P_i, $1\leqq i\leqq n$, Theorem 5.2 yields the existence of $>n\cdot\frac{1}{2}c_2 y(n-1)/c_{17}=c_{18}\cdot y\cdot n^2$ distinct circles each of them containing at least three but not more than c_{17} points P_j. ▮

Let there be given a point-set S and a hyperplane H. We say that H is an *S-hyperplane* if $H\cap S$ spans S.

Theorem 5.4. *Let there be given an n-element point-set in the Euclidean r-space. Then either*

(α) *some $(r-1)$-dimensional hyperplane contains more than $c_r'\cdot n$ points of S, or*

(β) *the number of distinct $(r-1)$-dimensional S-hyperplanes exceeds $c_r''\cdot n^r$.*

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* A simpler direct proof can be extracted from Kelly–Moser [9].

** A result of Kelly and Moser [9] yields $c_{17}=4$.

Proof. We show the validity of the *stronger* statement below.

Statement. Let there be given *n not necessarily distinct* points $P_1, P_2, \dots, P_n$ in the Euclidean r-space ($r \geqq 2$). Assume that $|\{P_1, P_2, \dots, P_n\}| > \varepsilon \cdot n$, i.e., there are more than $\varepsilon \cdot n$ distinct points among them. Then either

(α) some $(r-1)$-dimensional hyperplane contains $> c_r'(\varepsilon) \cdot n$ distinct points of $\{P_1, P_2, \dots, P_n\}$, or

(β) the number of distinct $(r-1)$-dimensional $\{P_1, P_2, \dots, P_n\}$-hyperplanes containing less than $c_r'''(\varepsilon) \cdot n$ not necessarily distinct points of $P_1, \dots, P_n$ exceeds $c_r''(\varepsilon) \cdot n^r$.

We proceed by induction on r. We start with the simplest case $r=2$. Let $|\{P_1, P_2, \dots, P_n\}| = N > \varepsilon \cdot n$. From Theorem 5.2 it follows that, either

(α) some straight line contains $\geqq N/2$ distinct points of $\{P_1, P_2, \dots, P_n\}$, or

(β) the number of connecting lines containing less than c_{17} distinct points of $\{P_1, P_2, \dots, P_n\}$ exceeds $\frac{1}{2} c_2 \cdot \frac{N}{2} \cdot N$.

Let $m(P_i) =$ multiplicity of $P_i = |\{j \in [1, n]\colon P_j = P_i\}|$.

Let us introduce the set

$$A(M) = \{P_i\colon m(P_i) \geqq M\}.$$

Since $|A(M)| \cdot M \leqq n$, we have $|A(M)| \leqq n/M$. Thus, the number of distinct connecting lines passing through some $P_i \in A(M)$ is at most $|A(M)| \cdot N \leqq n \cdot N/M < N^2 \cdot (\varepsilon M)^{-1}$. If $M \geqq M_0(\varepsilon)$ then $N^2(\varepsilon M)^{-1} < \frac{1}{8} c_2 N^2$, and we conclude that the number of connecting lines containing less than $c_{17}M$ points of $P_1, \dots, P_n$ exceeds $\frac{1}{8} c_2 \cdot N^2$. This completes the case $r=2$.

Now assume that the statement is true for $r-1 \geqq 2$. Then for every point P_i choose an $(r-1)$-dimensional hyperplane H_i such that $P_i \notin H_i$ and the straight line $P_i P_j$ *does* intersect H_i if $j \neq i$. Applying *central projection* onto H_i with center P_i, the projected image $P_1', P_2', \dots, P_{i-1}', P_{i+1}', \dots, P_n'$ of $P_1, P_2, \dots, P_{i-1}, P_{i+1}, \dots, P_n$ contains more than $c_1(\varepsilon n - 1)$ distinct points. Indeed, it is an immediate consequence of Theorem 1.1. Now applying the induction hypothesis for $P_1', P_2', \dots, P_{i-1}', P_{i+1}', \dots, P_n'$, and summing for $i = 1, 2, \dots, n$, one can easily complete the induction step. Theorem 5.4 follows. ∎

Finally, we mention an application of Theorem 1.5 to the 3-dimensional "unit distances" problem: Find bounds on $h(n) =$ the maximal number of unit distances determined by n points in the Euclidean 3-space.

Erdős [4] observed that

$$n^{4/3} \cdot \log\log n < h(n) < n^{5/3}.$$

Using Theorem 1.5 we shall give a "non-trivial" upper bound to $h(n)$.

Theorem 5.5. *Let n be sufficiently large. Then $h(n) < n^{5/3-c}$ where $c > 0$, independent of n.*

Remark. Replacing Theorem 1.5 with the Szemerédi—Trotter estimate mentioned in Section 1, our argument yields $h(n) < n^{39/24 + o(1)}$.

Proof. Let there be given a set $S=\{P_1, P_2, \ldots, P_n\}$ of n points in 3-space. Denote by $F(P)$ the unit sphere with center at P. Set

$$S(i,j)=\{P\in S\colon \{P_i, P_j\}\subset F(P)\}.$$

Observe that the points in $S(i,j)$ lie on a *circle* $R(i,j)$, i.e., $S(i,j)=S\cap R(i,j)$. Simple geometric consideration shows that if $(i,j)\neq(k,l)$ then the circles $R(i,j)$ and $R(k,l)$ are *distinct*.

Let $h_i=|F(P_i)\cap S|$, $1\leqq i\leqq n$. Counting the number of triples (P, P', P'') satisfying properties $P, P', P''\in S$ and $\{P', P''\}\subset F(P)$ in two different ways we get

$$\sum_{1\leqq i<j\leqq n}|S(i,j)|=\sum_{i=1}^{n}\binom{h_i}{2}. \tag{25}$$

To estimate the left-hand side of (25) from above, we introduce the number $T(p)=T(S,p)$ of distinct circles containing at least p points of S. We call these circles as p-circles of S. Simple averaging argument gives that for some point P_i there are at least $p\cdot T(p)/n$ p-circles of S passing through P_i. Applying *inversion* with center P_i we obtain the existence of $p\cdot T(p)/n$ distinct straight lines each of them passing through at least $p-1$ points of the inverted image of S. Consequently, $p\cdot T(p)/n\leqq t(n-1, p-1)$. We have therefore from Theorem 1.5 and inequality (2) in Lemma 2.1 after easy calculation that

$$T(p)\leqq c'\frac{n^3}{p^{3+\delta}},\qquad \delta=\frac{1}{20},\quad \text{for all}\quad 3\leqq p\leqq(2n)^{1/2}, \tag{26}$$

and

$$T(p)\leqq 2\cdot\frac{n^2}{p^2}\quad\text{for}\quad (2n)^{1/2}<p\leqq n. \tag{27}$$

From (26) we obtain

$$\sum_{q\leqq|S(i,j)|\leqq(2n)^{1/2}}|S(i,j)|=\sum_{l\geqq1}\ \sum_{\substack{2^{l-1}q\leqq|S(i,j)|<2^l q\\ |S(i,j)|\leqq(2n)^{1/2}}}|S(i,j)| \leqq \sum_{l\geqq1}2^l\cdot q\cdot c'\frac{n^3}{(2^{l-1}\cdot q)^{3+\delta}}\leqq c''\cdot\frac{n^3}{q^{2+\delta}}. \tag{28}$$

By (27) we get

$$\sum_{(2n)^{1/2}<|S(i,j)|\leqq n}|S(i,j)|=\sum_{l\geqq1}\ \sum_{2^{l-1}\cdot(2n)^{1/2}<|S(i,j)|\leqq 2^l\cdot(2n)^{1/2}}|S(i,j)|\leqq c'''\cdot n^2. \tag{29}$$

Summarising, by (28) and (29) we have

(30)
$$\sum_{1\leqq i<j\leqq n}|S(i,j)|=\sum_{|S(i,j)|\leqq q}|S(i,j)|+\sum_{q\leqq|S(i,j)|\leqq(2n)^{1/2}}|S(i,j)|+\sum_{(2n)^{1/2}<|S(i,j)|\leqq n}|S(i,j)|$$
$$<\binom{n}{2}\cdot q+c''\frac{n^3}{q^{2+\delta}}+c'''\cdot n^2<\frac{1}{3}n^{2+\frac{1}{3}-2c}$$

where $q=n^{\frac{1}{3+\delta}}$, $c=\dfrac{\delta}{6(3+\delta)}-\varepsilon$ and $n>n_0(\varepsilon)$.

On the other hand, clearly

$$\sum_{i=1}^{n}\binom{h_i}{2} \geqq n\binom{\sum_{i=1}^{n} h_i/n}{2} \geqq \frac{\left(\sum_{i=1}^{n} h_i\right)^2}{3n}. \tag{31}$$

Comparing (25), (30) and (31) we conclude that

$$h = \sum_{i=1}^{n} h_i < n^{5/3-c},$$

that is, the number h of unit distances determined by S is less than $n^{5/3-c}$ with $c=\frac{\delta}{6(2+\delta)}-\varepsilon$ if $n>n_0(\varepsilon)$. ∎

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